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ELASTIC ANISOTROPY OF SHORT-FIBRE REINFORCED COMPOSITES

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Abstract—Short-fibre reinforced composites are attractive because of their ease of fabrication and relatively low cost. They consist of partially aligned short fibres in a continuous matrix material, the orientation of the fibres depending on the processing conditions employed. In this report a theoretical treatment of the elastic anisotropy of a short-fibre reinforced composite resulting from a partial orientation of the fibres is presented. The elastic stiffnesses of the composite may be calculated in terms of the coefficients W_{lmn} in an expansion of the fibre-orientation distribution function in generalized Legendre functions. Since the elastic stiffness tensor is of fourth rank, it depends only on the coefficients W_{lmn} for $l \leqslant 4$. These coefficients may be determined from the angular variation of the ultrasonic velocity. This allows the fibre-orientation distribution to be plotted. Since the anisotropy in strength properties originates from the preferred orientation of fibres, the prediction of the fibre orientation distribution function will have an important application in the failure analysis of these materials.

1. INTRODUCTION

Short-fibre reinforced composites are attractive because of their ease of fabrication and relatively low cost. They consist of partially aligned short fibres in a continuous matrix material, the orientation of the fibres depending on the processing conditions employed. Templeton (1990) has recently studied the parameters which influence the strength of short-fibre reinforced composites produced by injection moulding. It was found that the fibre volume fraction and orientation play a more important role in controlling the strength than the other parameters considered which included fibre and resin strength, fibre critical length, average fibre length and a bonding efficiency factor. In injection moulding the orientation of the fibres is largely determined by the flow rheology and this strongly influences the mechanical properties of the composite, which are stronger and stiffer in the direction of maximum orientation. The purpose of the present paper is to examine the sensitivity of the elastic stiffness tensor to the fibre-orientation distribution. The fibre-orientation distribution function is defined in Section 2. The elastic stiffness tensor is calculated in Section 3 in terms of the coefficients W_{lmn} occurring in an expansion of this function in generalized Legendre functions. A similar use of the fibre-orientation distribution function was made by Ferrari and Johnson (1989) although no numerical results were obtained. It is shown here that the elastic stiffness tensor of the composite may be expressed in terms of the W_{lmn} for $l \leqslant 4$ and three parameters a_1, a_2 and a_3 characterizing the anisotropy of a composite with perfectly aligned fibres. These parameters are evaluated in Section 4.

Since the processing conditions vary from manufacturer to manufacturer it is of interest to be able to determine the fibre-orientation distribution experimentally. This is usually done by image analysis of photomicrographs taken from thin sections of the material, but this is both time consuming and destructive. Non-destructive techniques for determining the fibre-orientation distribution would be preferable. In this report the possibility of using ultrasonic velocity measurements for this purpose is examined. In elastically isotropic materials, the ultrasonic velocities are independent of the propagation direction and, in the case of shear waves, the direction of polarization. An anisotropic fibre-orientation distribution will remove this isotropic behaviour. The resulting anisotropy in the ultrasonic velocities may therefore be used to characterize the fibre orientation distribution. In Section 5, the explicit expressions for the elastic stiffness tensor in terms of the expansion coefficients

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of the fibre-orientation distribution function are used to calculate the ultrasonic velocities. The expressions that result enable the inversion of the measured ultrasonic velocities for the coefficients W_{lmn} of the fibre-orientation distribution function for $l \leqslant 4$. This corresponds to the specification of the orientation distribution by its first few moments which may be used in equations of the type given by Templeton (1990) to predict the strength of the composite.

2. THE FIBRE-ORIENTATION DISTRIBUTION FUNCTION

As a result of the processing conditions, the fibres in a short-fibre reinforced composite will be partially aligned. To model the effect of a preferred orientation of fibres it is convenient to introduce a set of axes $OX_1X_2X_3$ with origin at the centre of the fibre and OX_3 along the fibre axis. Let the fourth-order effective elastic stiffness tensor of the composite be denoted by C^*_{ijkl}. Consider first the case in which the axes $OX_1X_2X_3$ for all fibres are aligned. If the C^*_{ijkl} in this case are denoted by C^a_{ijkl}, then, for fibres with circular cross-section, the non-zero C^a_{ijkl} are $C^a_{11} = C^a_{22}$, C^a_{33}, $C^a_{12} = C^a_{21}$, $C^a_{23} = C^a_{32} = C^a_{31} = C^a_{13}$, $C^a_{44} = C^a_{55}$ and $C^a_{66} = (C^a_{11} - C^a_{12})/2$ in the Voigt (two-index) notation. The anisotropy of C^a_{ijkl} may therefore be completely specified in this case by three anisotropy parameters a_1, a_2 and a_3 (Sayers, 1990) defined by:

$$a_1 = C^a_{11} + C^a_{33} - 2C^a_{13} - 4C^a_{44}, \tag{1}$$

$$a_2 = C^a_{11} - 3C^a_{12} + 2C^a_{13} - 2C^a_{44}, \tag{2}$$

$$a_3 = 4C^a_{11} - 3C^a_{33} - C^a_{13} - 2C^a_{44}. \tag{3}$$

In general, the fibres will not be perfectly aligned and a quantitative description of the elastic anisotropy requires a knowledge of the orientation distribution of fibres. The orientation of a fibre with elliptical cross-section with principal axes $OX_1X_2X_3$ with respect to a set of axes $Ox_1x_2x_3$ fixed in the composite may be specified by three Euler angles ψ, θ and ϕ. The orientation distribution of fibres is then given by the fibre orientation distribution function $W(\xi, \psi, \phi)$ where $\xi = \cos\theta$, θ being the angle between OX_3 and Ox_3. $W(\xi, \psi, \phi)\,\mathrm{d}\xi\,\mathrm{d}\psi\,\mathrm{d}\phi$ gives the fraction of fibres between ξ and $\xi + \mathrm{d}\xi$, ψ and $\psi + \mathrm{d}\psi$, and ϕ and $\phi + \mathrm{d}\phi$. Clearly,

$$\int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{1} W(\xi, \psi, \phi)\,\mathrm{d}\xi\,\mathrm{d}\psi\,\mathrm{d}\phi = 1. \tag{4}$$

3. ELASTIC ANISOTROPY DUE TO PARTIAL FIBRE ALIGNMENT

In the following, an approximate treatment of the fibre interactions will be employed, in which the effective elastic constants of the composite are calculated by summing up the contributions to the stiffness from fibres of all orientations, calculated as if all fibres in the composite had the same orientation as the fibre under consideration. Since the interaction between fibres decreases with increasing separation, this approximation is expected to be best for composites in which the orientations of neighbouring fibres are strongly correlated, as is the case in composites produced by injection moulding. The approximation is expected to be worst for randomly orientated fibres with no correlation between the orientations of neighbouring fibres.

The elastic stiffness of the composite may therefore be calculated in this approximation from the C^a_{ijkl} and the fibre-orientation distribution function as follows. If $T_{ijklmnpq}$ is given by the transformation rule for tensors of rank four, i.e.

$$T_{ijklmnpq} = \left(\frac{\partial x_i}{\partial X_m}\right)\left(\frac{\partial x_j}{\partial X_n}\right)\left(\frac{\partial x_k}{\partial X_p}\right)\left(\frac{\partial x_l}{\partial X_q}\right). \tag{5}$$

then, taking into account the orientation distribution of fibres, the elastic stiffnesses of the composite C^*_{ijkl} are given by

$$C^*_{ijkl} = \bar{T}_{ijklmnpq} C^a_{mnpq}, \tag{6}$$

where

$$\bar{T}_{ijklmnpq} = \int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{1} T_{ijklmnpq}(\xi,\psi,\phi) W(\xi,\psi,\phi)\,d\xi\,d\psi\,d\phi. \tag{7}$$

These integrals may be evaluated by expanding the fibre orientation distribution function $W(\xi,\psi,\phi)$ as a series of generalized spherical harmonics and using the orthogonality relations between these functions (Morris, 1969). Since the elastic stiffness tensor is of fourth rank it depends only on the coefficients W_{lmn} of the expansion of $W(\xi,\psi,\phi)$ for $l \leqslant 4$. If the fibres have an elliptical cross-section and their orientation distribution is orthotropic with symmetry axes coincident with the reference axes $Ox_1x_2x_3$, the non-zero W_{lmn} are all real and are restricted to even values of l, m and n. For fibres with a circular cross-section, $W_{lmn} = 0$ unless $n = 0$. The elastic stiffnesses are therefore determined in this case by W_{200}, W_{220}, W_{400}, W_{420} and W_{440} and the three anisotropy factors a_1, a_2 and a_3 defined above (Sayers, 1990). The a_i will be calculated in Section 4.

For fibres with circular cross-section, the equations for the C^*_{ijkl} are found to be:

$$C^*_{11} = \lambda^* + 2\mu^* + \frac{8\sqrt{10}}{105}\pi^2 a_3(W_{200} - \sqrt{6}W_{220}) + \frac{4\sqrt{2}}{35}\pi^2 a_1\left(W_{400} - \frac{2\sqrt{10}}{3}W_{420} + \frac{\sqrt{70}}{3}W_{440}\right), \tag{8}$$

$$C^*_{22} = \lambda^* + 2\mu^* + \frac{8\sqrt{10}}{105}\pi^2 a_3(W_{200} + \sqrt{6}W_{220}) + \frac{4\sqrt{2}}{35}\pi^2 a_1\left(W_{400} + \frac{2\sqrt{10}}{3}W_{420} + \frac{\sqrt{70}}{3}W_{440}\right), \tag{9}$$

$$C^*_{33} = \lambda^* + 2\mu^* - \frac{16\sqrt{2}}{105}\pi^2(\sqrt{5}a_3 W_{200} - 2a_1 W_{400}), \tag{10}$$

$$C^*_{12} = \lambda^* - \frac{8\sqrt{10}}{315}\pi^2(7a_2 - a_3)W_{200} + \frac{4\sqrt{2}}{105}\pi^2 a_1(W_{400} - \sqrt{70}W_{440}), \tag{11}$$

$$C^*_{31} = \lambda^* + \frac{4\sqrt{10}}{315}\pi^2(7a_2 - a_3)(W_{200} + \sqrt{6}W_{220}) - \frac{16\sqrt{2}}{105}\pi^2 a_1(W_{400} - \sqrt{5/2}W_{420}), \tag{12}$$

$$C^*_{23} = \lambda^* + \frac{4\sqrt{10}}{315}\pi^2(7a_2 - a_3)(W_{200} - \sqrt{6}W_{220}) - \frac{16\sqrt{2}}{105}\pi^2 a_1(W_{400} + \sqrt{5/2}W_{420}), \tag{13}$$

$$C^*_{44} = \mu^* - \frac{2\sqrt{10}}{315}\pi^2(7a_2 + 2a_3)(W_{200} - \sqrt{6}W_{220}) - \frac{16\sqrt{2}}{105}\pi^2 a_1(W_{400} + \sqrt{5/2}W_{420}), \tag{14}$$

$$C^*_{55} = \mu^* - \frac{2\sqrt{10}}{315}\pi^2(7a_2+2a_3)(W_{200}+\sqrt{6}W_{220}) - \frac{16\sqrt{2}}{105}\pi^2 a_1(W_{400}-\sqrt{5/2}W_{420}). \quad (15)$$

$$C^*_{66} = \mu^* + \frac{4\sqrt{10}}{315}\pi^2(7a_2+2a_3)W_{200} + \frac{4\sqrt{2}}{105}\pi^2 a_1(W_{400}-\sqrt{70}W_{400}). \quad (16)$$

Here λ^* and μ^* are given by $15\lambda^* = C^a_{11}+C^a_{33}+5C^a_{12}+8C^a_{13}-4C^a_{44}$ and $30\mu^* = 7C^a_{11}+2C^a_{33}-5C^a_{12}-4C^a_{13}+12C^a_{44}$. In principle, eqns (8)–(16) enable the determination of the W_{lm0} for $l \leqslant 4$ from the measured elastic stiffness tensor of the composite provided the a_i can be determined. This then allows the fibre-orientation distribution function to be plotted. The a_i are calculated in Section 4. Conversely, if the a_i are known, the expansion coefficients W_{200}, W_{220}, W_{400}, W_{420} and W_{440} are sufficient to calculate the elastic stiffnesses for *any* orthotropic orientation distribution of fibres. Note that in the case of a transversely isotropic distribution of fibres with symmetry axis along Ox_3, $W_{220} = W_{420} = W_{440} = 0$ and the elastic stiffnesses are determined by only two expansion coefficients W_{200} and W_{400} of the fibre-orientation distribution function.

4. CALCULATION OF THE a_i

Following Willis (1983, 1984), consider a two-phase composite occupying a domain V with displacement or traction components, or some combination of these, prescribed on the exterior boundary S. The basic elastostatic problem is to solve the equilibrium conditions:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad (17)$$

subject to the boundary conditions. Here σ is the stress tensor which is related, symbolically, to the strain tensor ε by $\sigma = C\varepsilon$. In components this reads $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$. C is the elastic stiffness tensor which varies with position in the composite. The composites of interest consist of a matrix with elastic stiffness tensor $C^{(1)}$ in which are embedded inclusions of elastic stiffness tensor $C^{(2)}$ whose centres are distributed throughout the composite according to some stochastic process. The stress, strain and displacement fields depend on the location of the inclusions and the objective is to find their ensemble averages $\langle\sigma\rangle$, $\langle\varepsilon\rangle$ and $\langle u\rangle$. To treat this problem, Willis (1983, 1984) introduces a homogeneous "comparison medium" with elastic stiffness tensor $C^{(0)}$. Defining the stress polarization tensor τ by $\tau = (C-C^{(0)})\varepsilon$, it follows that $\sigma = C^{(0)}\varepsilon+\tau$. Equation (17) then gives

$$C^{(0)}_{ijkl}\frac{\partial \varepsilon_{kl}}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} = 0. \quad (18)$$

For a uniform elastic medium with stiffness tensor $C^{(0)}_{ijkl}$, the elastic Green's function $g^{(0)}_{ij}(x, x')$ satisfies the equation:

$$C^{(0)}_{ijkl}\frac{\partial^2 g^{(0)}_{kp}(x, x')}{\partial x_l\, \partial x_j} = -\delta_{ip}\delta(x-x'). \quad (19)$$

A knowledge of this function allows the strain field in the composite to be written as an integral equation:

$$\varepsilon_{ij}(x) = \varepsilon^{(0)}_{ij} - \int G^{(0)}_{ijkl}(x, x')\tau_{kl}(x')\,\mathrm{d}x'. \quad (20)$$

Here, $\varepsilon^{(0)}$ is the solution of the given boundary value problem for the comparison medium and $G^{(0)}$ is an operator related to the Green's function $g^{(0)}(x, x')$:

$$G^{(0)}_{ijkl}(x,x') = \left.\frac{\partial^2 g^{(0)}_{jk}(x,x')}{\partial x_i \partial x'_l}\right|_{(ij)(kl)}, \tag{21}$$

where the suffix (ij) implies symmetrization with respect to these indices. For a composite consisting of aligned ellipsoidal inclusions of identical shape Willis (1983, 1984) solves this equation using a "closure assumption" which Willis identifies as Lax's quasi-crystalline approximation in multiple scattering theory (Lax, 1952). This gives the following approximation for the effective elastic stiffness of a composite with aligned inclusions:

$$C^a = C^{(1)} + v_2[(C^{(2)} - C^{(1)})^{-1} + (1-v_2)P]^{-1}, \tag{22}$$

where $P = SC^{(1)^{-1}}$, S is Eshelby's tensor (Eshelby, 1957) and v_i is the volume fraction of the ith phase. Defining $T = [I + P(C^{(2)} - C^{(1)})]^{-1}$, Benveniste (1990) shows that this equation may be written in the form:

$$C^a = C^{(1)} + v_2(C^{(2)} - C^{(1)})T[v_1 I + v_2 T]^{-1}. \tag{23}$$

Benveniste (1990) shows further that this result is identical to that obtained using the method of Mori and Tanaka (1973) for the case of aligned ellipsoidal inclusions. The components of C^a may therefore be derived from the work of Tandon and Weng (1984) and Zhao *et al.* (1989) who used the method of Mori and Tanaka (1973) to derive the engineering elastic moduli for a composite with aligned inclusions. For the case of spheroidal inclusions with axes aligned along the Ox_3 direction, these constants are the longitudinal Young's modulus E^a_{33}, the transverse Young's modulus E^a_{11}, the in-plane shear modulus μ^a_{31}, the out-of-plane shear modulus μ^a_{12}, the plane-strain bulk modulus K^a_{12} and the major Poisson's ratio v^a_{31}. The C^a_{ij} may be obtained from these results using the following relations:

$$C^a_{11} = C^a_{22} = \mu^a_{12} + K^a_{12}, \tag{24}$$

$$C^a_{33} = E^a_{33} + 4v^{a^2}_{31}K^a_{12}, \tag{25}$$

$$C^a_{12} = -\mu^a_{12} + K^a_{12}, \tag{26}$$

$$C^a_{31} = C^a_{23} = 2v^{a^2}_{31}K^a_{12}, \tag{27}$$

$$C^a_{44} = C^a_{55} = \mu^a_{31}, \tag{28}$$

$$C^a_{66} = \tfrac{1}{2}(C^a_{11} - C^a_{12}) = \mu^a_{12}. \tag{29}$$

With the choice of axes specified above, the moduli derived by Tandon and Weng (1984) are listed below:

$$\frac{E^a_{33}}{E^{(1)}} = \frac{1}{1 + v_2(A_1 + 2v^{(1)}A_2)/A}, \tag{30}$$

$$\frac{E^a_{11}}{E^{(1)}} = \frac{1}{1 + v_2[-2v^{(1)}A_3 + (1-v^{(1)})A_4 + (1+v^{(1)})A_5 A]/2A}, \tag{31}$$

$$\frac{\mu^a_{31}}{\mu^{(1)}} = 1 + \frac{v_2}{\dfrac{\mu^{(1)}}{\mu^{(2)} - \mu^{(1)}} + 2(1-v_2)S_{3131}}, \tag{32}$$

$$\frac{\mu_{12}^{a}}{\mu^{(1)}} = 1 + \frac{v_2}{\dfrac{\mu^{(1)}}{\mu^{(2)}-\mu^{(1)}} + 2(1-v_2)S_{1212}}, \tag{33}$$

$$\frac{K_{12}^{a}}{\bar{K}^{(1)}} = \frac{(1+v^{(1)})(1-2v^{(1)})}{1-v^{(1)}(1+2v_{31}^{a})+v_2[2(v_{31}^{a}-v^{(1)})A_3+(1-v^{(1)}(1+2v_{31}^{a}))A_4]/A}. \tag{34}$$

Expressions for A, A_1, A_2, A_3, A_4 and A_5 have been given by Tandon and Weng (1984). $\bar{K}^{(1)}$ is the plane strain bulk modulus of the matrix, and is given by $\bar{K}^{(1)} = \lambda^{(1)}+\mu^{(1)}$. The major Poisson's ratio v_{31}^{a} is not an independent modulus but is related to the others by

$$v_{31}^{a^2} = \frac{E_{33}^{a}}{E_{11}^{a}} - \frac{E_{33}^{a}}{4}\left(\frac{1}{\mu_{12}^{a}}+\frac{1}{K_{12}^{a}}\right). \tag{35}$$

Zhao *et al.* (1989) have obtained the following explicit expression for v_{31}^{a}:

$$v_{31}^{a} = v^{(1)} - v_2\frac{v^{(1)}(A_1+2v^{(1)}A_2)+(A_3-v^{(1)}A_4)}{A+v_2(A_1+2v^{(1)}A_2)}. \tag{36}$$

These expressions were evaluated for the case of glass fibres in an epoxy matrix, for which $E^{(1)} = 2.76$ GPa, $v^{(1)} = 0.35$, $E^{(2)} = 72.4$ GPa, $v^{(2)} = 0.2$ (Tandon and Weng, 1984). Figure 1 shows the variation of $C_{11}^{a}/C_{11}^{(1)}$, $C_{33}^{a}/C_{33}^{(1)}$, $C_{12}^{a}/C_{12}^{(1)}$, $C_{31}^{a}/C_{31}^{(1)}$, $C_{44}^{a}/C_{44}^{(1)}$ and $C_{66}^{a}/C_{66}^{(1)}$ with volume fraction v_2 for various aspect ratios α. Here $C_{11}^{(1)} = C_{33}^{(1)} = \lambda^{(1)}+2\mu^{(1)}$, $C_{12}^{(1)} = C_{31}^{(1)} = \lambda^{(1)}$ and $C_{44}^{(1)} = C_{66}^{(1)} = \mu^{(1)}$. C_{66}^{a} is not independent but is given by $C_{66}^{a} = (C_{11}^{a}-C_{12}^{a})/2$. C_{33}^{a} and C_{31}^{a} are seen to be sensitive to the aspect ratio α of the inclusions whilst C_{11}^{a}, C_{12}^{a}, C_{44}^{a} and C_{66}^{a} are relatively insensitive.

It is seen in Fig. 1 that as the fibre volume fraction approaches unity, the elastic constants of the composite become equal to the isotropic elastic constants of the fibres. In a real composite material, however, the fibre volume fraction is limited by the maximum packing density and the theory will therefore not be applicable after this volume fraction is reached. Figure 2 shows the anisotropy factors a_1, a_2 and a_3 defined by eqns (1)–(3) as a function of volume fraction v_2 for various aspect ratios α. The maxima and minima seen in these figures at high volume fraction are artefacts of the theory resulting from the elastic constants of the composite becoming equal to the isotropic elastic constants of the fibres at high volume fractions. The theory is therefore inapplicable for volume fractions much above 70%.

A knowledge of the a_i allows the elastic stiffnesses C_{ijkl}^{*} of the composite to be determined for *any* orthotropic orientation distribution of fibres by using eqns (8)–(16) and the first few coefficients W_{lmn} in an expansion of the orientation distribution function in generalized spherical harmonics. The W_{lmn} are given by

$$W_{lmn} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_{-1}^{1} W(\xi,\psi,\phi)Z_{lmn}(\xi)e^{im\psi}e^{in\phi}\,d\xi\,d\psi\,d\phi. \tag{37}$$

Here the $Z_{lmn}(\xi)$ are the generalized Legendre functions defined by Roe (1965). W_{lmn} therefore represents the value of a polynomial of trigonometrical functions of θ, ψ and ϕ averaged over all fibre orientations. The use of a limited number of W_{lmn} therefore corresponds to the specification of the orientation distribution by its few moments. These may then be used in equations of the type given by Templeton (1990) to predict the strength of the composite. It is interesting to note the values of W_{200} and W_{220} for the cases of complete fibre alignment. For complete alignment of fibre axes along Ox_3, $W_{200} = \sqrt{10}/8\pi^2 = 0.04005$, $W_{220} = 0$ whilst for the case of fibre axes being randomly orientated in the x_1x_2 plane, $W_{200} = -\sqrt{10}/16\pi^2 = -0.02003$, $W_{220} = 0$. Perfect alignment of fibres along Ox_1 corresponds to $W_{200} = -\sqrt{10}/16\pi^2 = -0.02003$,

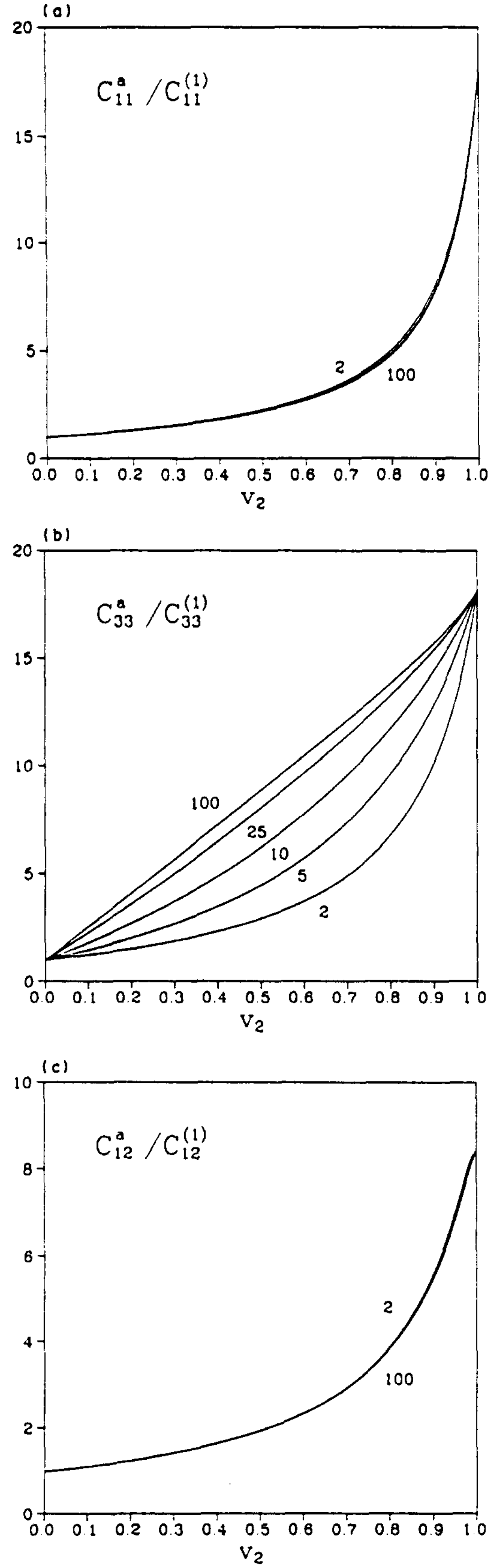
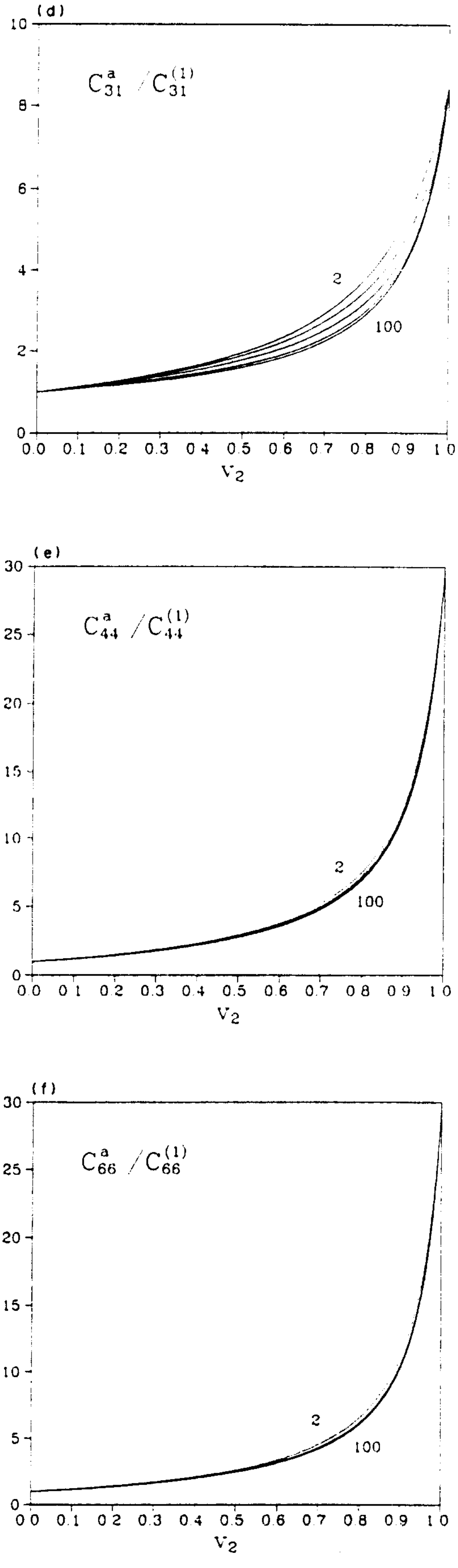


Fig. 1. The variation of (a) $C^a_{11}/C^{(1)}_{11}$, (b) $C^a_{33}/C^{(1)}_{33}$, (c) $C^a_{12}/C^{(1)}_{12}$, (d) $C^a_{31}/C^{(1)}_{31}$, (e) $C^a_{44}/C^{(1)}_{44}$ and (f) $C^a_{66}/C^{(1)}_{66}$ with volume fraction v_2 for aspect ratios $\alpha = 2, 5, 10, 25$ and 100 as indicated on the curves. The superscripts a and (1) denote the properties of the composite with perfectly aligned fibres and the matrix phase respectively. Where the curves are closely spaced only the curves for $\alpha = 2$ and 100 are labelled.



(d)
$C^a_{31}/C^{(1)}_{31}$
2
100
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
V_2
(e)
$C^a_{44}/C^{(1)}_{44}$
2
100
V_2
(f)
$C^a_{66}/C^{(1)}_{66}$
2
100
V_2

Fig. 1 (*contd.*)

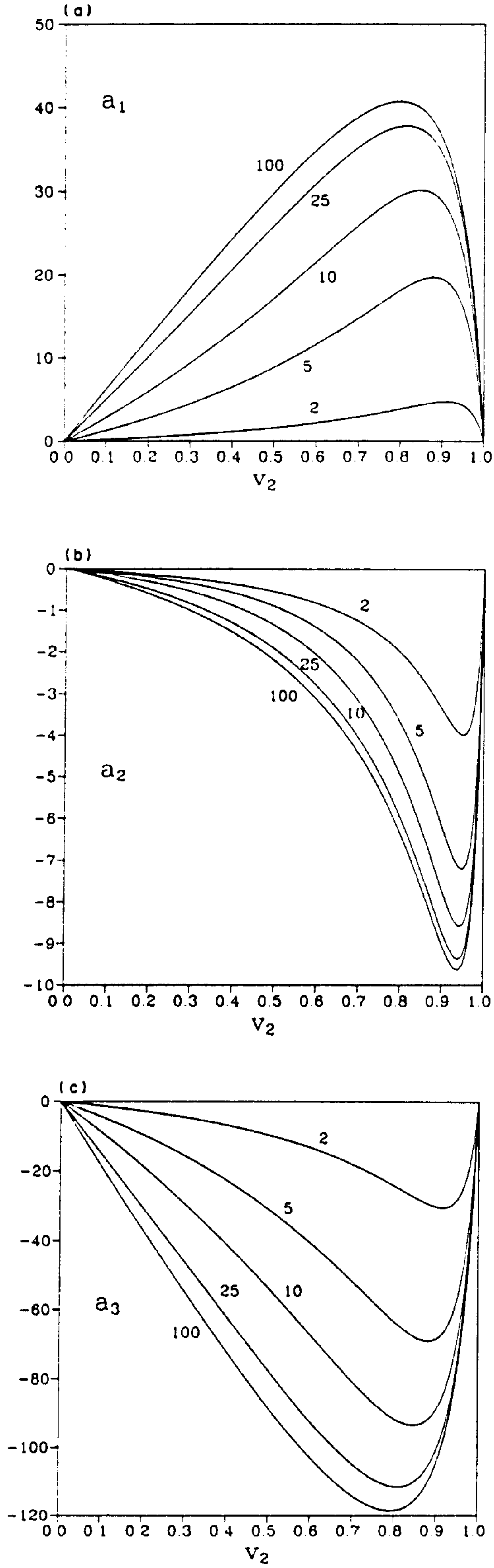


Fig. 2. The anisotropy factors (a) a_1, (b) a_2 and (c) a_3 defined by eqns (1)–(3) for aspect ratios $\alpha = 2, 5, 10, 25$ and 100 as indicated on the curves.

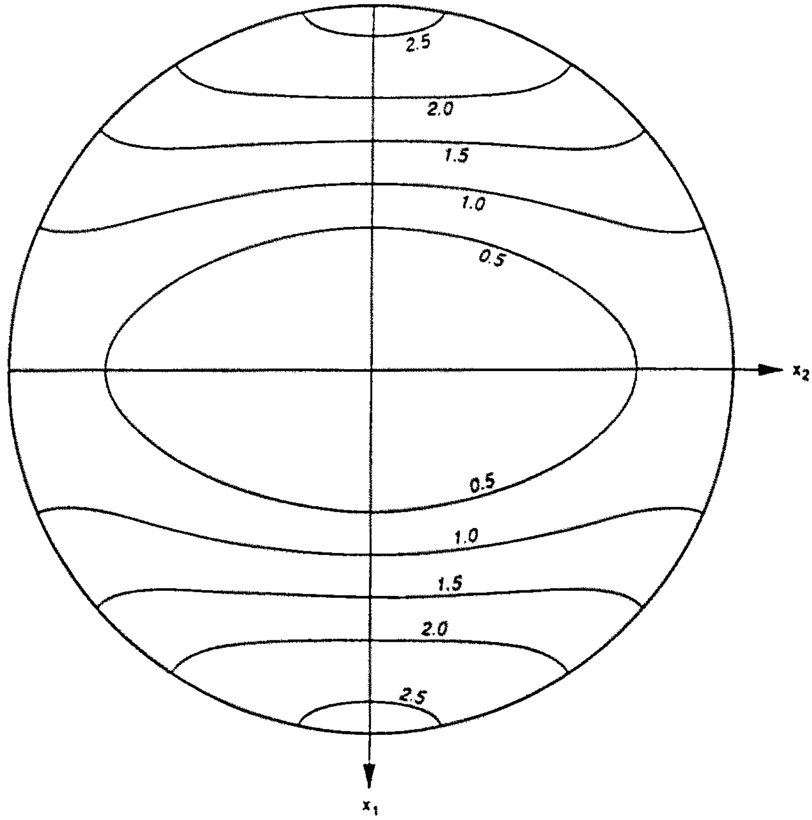


Fig. 3. Equal area projection of the orientation distribution of fibre axes for the cases $W_{200} = -0.0105$, $W_{220} = 0.00646$.

$W_{220} = \sqrt{15/16\pi^2} = 0.02453$ whilst for complete alignment along Ox_2, $W_{200} = -\sqrt{10/16\pi^2} = -0.02003$, $W_{220} = -\sqrt{15/16\pi^2} = -0.02453$.

As an example, Fig. 3 shows an equal area projection of the orientation distribution of fibre axes for the case $W_{200} = -0.0105$, $W_{220} = 0.00646$. The Ox_3 direction lies at the centre of this figure. A random distribution of fibres would correspond to a value of one everywhere in this figure. The peaks therefore correspond to a preferential orientation of fibre axes. It is seen that the case illustrated in Fig. 3 corresponds to most fibres lying in the x_1x_2 plane with a preferential alignment of fibres along Ox_1. The elastic stiffnesses can be calculated from eqns (8)–(16) and are plotted as a function of volume fraction in Fig. 4 for the case of glass fibres with aspect ratio $\alpha = 100$ in epoxy matrix.

5. ULTRASONIC ANISOTROPY DUE TO PARTIAL FIBRE ALIGNMENT

The ultrasonic wave velocities in the composite may be obtained as solutions of the Christoffel equations (Musgrave, 1970). If v_{ij} denotes the velocity of ultrasound propagating in the Ox_i direction with polarization in the Ox_j direction, the equations for the velocities are:

$$\rho v_{11}^2 = \lambda^* + 2\mu^* + 4\sqrt{2}\pi^2[2\sqrt{5}a_3(W_{200} - \sqrt{6}W_{220}) + a_1(3W_{400} - 2\sqrt{10}W_{420} + \sqrt{70}W_{440})]/105, \quad (38)$$

$$\rho v_{22}^2 = \lambda^* + 2\mu^* + 4\sqrt{2}\pi^2[2\sqrt{5}a_3(W_{200} + \sqrt{6}W_{220}) + a_1(3W_{400} + 2\sqrt{10}W_{420} + \sqrt{70}W_{440})]/105, \quad (39)$$

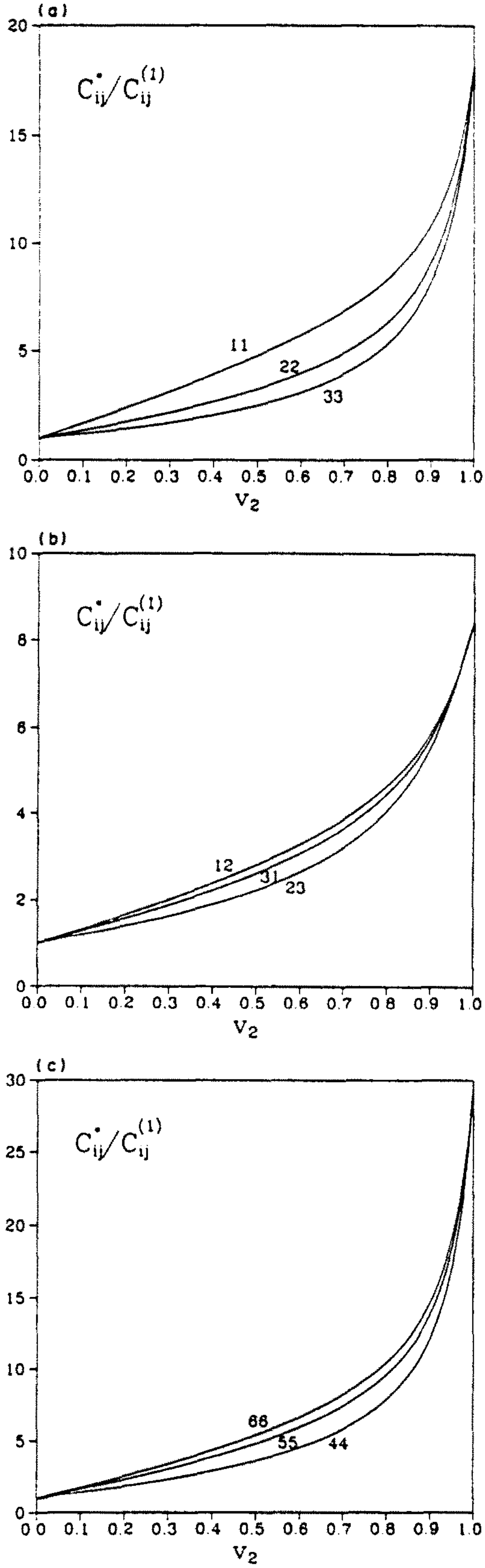


Fig. 4. The elastic stiffnesses $C^*_{ij}/C^{(1)}_{ij}$ for (a) ij = 11, 22 and 33, (b) ij = 12, 31 and 23, and (c) ij = 44, 55 and 66 calculated from eqns (8)–(16) plotted as a function of volume fraction for the case of glass fibres with aspect ratio $\alpha = 100$ and orientation parameters $W_{200} = -0.0105$, $W_{220} = 0.00646$ in an epoxy matrix. The superscipts * and (1) denote the properties of the composite and the matrix phase respectively. The values of ij are indicated on the curves.

$$\rho v_{33}^2 = \lambda^* + 2\mu^* - 16\sqrt{2}\pi^2[\sqrt{5}a_3 W_{200} - 2a_1 W_{400}]/105, \tag{40}$$

$$\rho v_{12}^2 = \rho v_{21}^2 = \mu^* + 4\sqrt{2}\pi^2[\sqrt{5}(7a_2 + 2a_3) W_{200} + 3a_1(W_{400} - \sqrt{70} W_{440})]/315, \tag{41}$$

$$\rho v_{23}^2 = \rho v_{32}^2 = \mu^* - 2\sqrt{2}\pi^2[\sqrt{5}(7a_2 + 2a_3)(W_{200} - \sqrt{6} W_{220}) + 24a_1(W_{400} + \sqrt{5/2} W_{420})]/315, \tag{42}$$

$$\rho v_{31}^2 = \rho v_{13}^2 = \mu^* - 2\sqrt{2}\pi^2[\sqrt{5}(7a_2 + 2a_3)(W_{200} + \sqrt{6} W_{220}) + 24a_1(W_{400} - \sqrt{5/2} W_{420})]/315. \tag{43}$$

Here ρ is the density of the composite and λ^* and μ^* are defined above.

The equations for off-axis propagation are more complicated and are not given here. They are easily obtained, however, from the Christoffel equations using eqns (8)–(16) and are sufficient to allow the W_{lm0} to be determined for $l \leqslant 4$.

6. CONCLUSION

In this paper a theoretical treatment of the elastic anisotropy of a short-fibre reinforced composite resulting from a partial orientation of the fibres is presented. The elastic stiffnesses of the composite may be calculated in terms of the coefficients W_{lmn} in an expansion of the fibre-orientation distribution function in generalized Legendre functions. Since the elastic stiffness tensor is of fourth rank, it depends only on the coefficients W_{lmn} for $l \leqslant 4$. These coefficients may be determined from the angular variation of the ultrasonic velocity. This allows the fibre-orientation distribution to be plotted. The inversion of the ultrasonic velocities for the coefficients W_{lmn} for $l \leqslant 4$ corresponds to the specification of the orientation distribution by its first few moments which may then be used in equations of the type given by Templeton (1990) to predict the strength of the composite. Since the anisotropy in strength properties originates from the preferred orientation of fibres, the prediction of the fibre-orientation distribution function will have an important application in the failure analysis of these materials.

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